

SEMESTER 1
APPLIED MATHEMATICS SOLVED PAPER – MAY 2017

Q.1(a) Prove that $\tanh^{-1}(\sin \theta) = \cosh^{-1}(\sec \theta)$ [3]

Ans : L.H.S = $\tanh^{-1}(\sin \theta)$

We know that , $\tanh^{-1}(x) = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

\therefore L.H.S = $\frac{1}{2} \log\left(\frac{1+\sin \theta}{1-\sin \theta}\right)$

R.H.S = $\cosh^{-1}(\sec \theta)$

We know that , $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$

\therefore R.H.S = $\log(\sec \theta + \sqrt{\sec^2 \theta - 1})$

$= \log\left(\frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta}\right)$ $\{\sqrt{\sec^2 \theta - 1} = \tan \theta = \frac{\sin \theta}{\cos \theta}\}$

$= \log\left(\frac{1+\sin \theta}{\cos \theta}\right)$

$= \log\left(\frac{1+\sin \theta}{\sqrt{1-\sin^2 \theta}}\right)$

$= \log\left(\frac{\sqrt{1+\sin \theta}}{\sqrt{1-\sin \theta}}\right)$

$= \frac{1}{2} \log\left(\frac{1+\sin \theta}{1-\sin \theta}\right)$

$\therefore \tanh^{-1}(\sin \theta) = \cosh^{-1}(\sec \theta)$

Hence Proved .

(b) Prove that the matrix $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary. [3]

Ans : Let $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

The matrix is unitary when $A.A^{\theta} = I$.

$$\therefore A^\theta = (\bar{A})^t = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}^t = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\begin{aligned} \therefore A.A^\theta &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\boxed{\therefore A.A^\theta = I}$$

The given matrix is unitary is proved.

(c) If $x=uv$ & $y=\frac{u}{v}$ prove that $JJ^1 = 1$ [3]

Ans: $x=uv$ and $y=\frac{u}{v}$

\therefore x and y are function of u and v .

$$\therefore u = \sqrt{xy} \quad \therefore v = \sqrt{\frac{x}{y}} \quad \dots\dots \{ \text{from given eqns} \}$$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = \frac{-u}{v} - \frac{u}{v} = \frac{-2u}{v} \quad \dots\dots(1)$$

$$J^1 = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \\ \frac{1}{2\sqrt{xy}} & \frac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix} = \frac{-\sqrt{x}}{2\sqrt{xy}} = \frac{-v}{2u} \quad \dots\dots(2)$$

$$\therefore JJ^1 = \frac{-2u}{v} \times \frac{-v}{2u} = 1$$

$$\boxed{\therefore JJ^1 = 1}$$

Hence Proved.

(d) If $z = \tan^{-1}\left(\frac{x}{y}\right)$, where $x=2t$, $y=1-t^2$, prove that $\frac{dz}{dt} = \frac{2}{1+t^2}$. [3]

Ans : $z = \tan^{-1}\left(\frac{x}{y}\right) \quad x=2t \quad \text{and} \quad y=1-t^2$

$\therefore z$ is the function of x and y & x and y are the functions of t .

$$z \rightarrow f(x,y) \rightarrow f(t)$$

$$\therefore z = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$$

Direct differentiate w.r.t t ,

$$\begin{aligned} \frac{dz}{dt} &= \frac{1}{1+\left(\frac{2t}{1-t^2}\right)^2} \times \frac{d}{dt}\left(\frac{2t}{1-t^2}\right) \\ &= \frac{2(1-t^2)^2}{(1-t^2)^2+4t^2} \times \left[t \cdot \frac{1}{(1-t^2)^2}(-2t) + \frac{1}{1-t^2} \times 1\right] \\ &= \frac{2(1-t^2)^2}{1+t^2} \times \frac{1}{(1-t^2)^2} \end{aligned}$$

$$\therefore \frac{dz}{dt} = \frac{2}{1+t^2}$$

Hence Proved.

(e) Find the n th derivative of $\cos 5x \cdot \cos 3x \cdot \cos x$. [4]

Ans : let $y = \cos 5x \cdot \cos 3x \cdot \cos x$

$$\begin{aligned} &= \frac{\cos(5x-3x) + \cos(5x+3x)}{2} \cdot \cos x \\ &= \frac{1}{2} [\cos 2x \cdot \cos x + \cos 8x \cdot \cos x] \\ y &= \frac{1}{4} [\cos 3x + \cos x + \cos 9x + \cos 7x] \end{aligned}$$

Take n th derivative,

$$n \text{th derivative of } \cos(ax + b) = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$$

$$y_n = \frac{1}{4} [9\cos\left(\frac{n\pi}{2} + 3x\right) + \cos\left(\frac{n\pi}{2} + x\right) + 81\cos\left(\frac{n\pi}{2} + 9x\right) + 49\cos\left(\frac{n\pi}{2} + 7x\right)]$$

(f) Evaluate : $\lim_{x \rightarrow 0} (x)^{\frac{1}{1-x}}$ [4]

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Ans: Let $L = \lim_{x \rightarrow 0} (x)^{\frac{1}{1-x}}$

Take log on both the sides,

$$\therefore \log L = \lim_{x \rightarrow 0} \frac{\log x}{1-x}$$

Apply L'Hospital rule ,

$$\therefore \log L = \lim_{x \rightarrow 0} \frac{1}{x}$$

$$= 0$$

$$\therefore L = e^0 = 1$$

Q.2(a) Find all values of $(1 + i)^{1/3}$ & show that their continued

Product is $(1+i)$.

[6]

Ans: let $x = (1 + i)^{1/3}$

$$\therefore x^3 = 1 + i = \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right)$$

$$\therefore x^3 = \sqrt{2} \left[\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right]$$

Add period $2k\pi$,

$$x^3 = \sqrt{2} \left[\cos \left(\frac{\pi}{4} + 2k\pi \right) + i \sin \left(\frac{\pi}{4} + 2k\pi \right) \right]$$

By applying De Moivre's theorem,

$$x = 2\sqrt{2} \left[\cos \frac{1}{3} \left(\frac{\pi}{4} + 2k\pi \right) + i \sin \frac{1}{3} \left(\frac{\pi}{4} + 2k\pi \right) \right]$$

where $k = 0, 1, 2$.

Roots are :

Put $k=0$ $x_0 = 2\sqrt{2} e^{i\frac{\pi}{12}}$

Put $k=1$ $x_1 = 2\sqrt{2} e^{i\frac{9\pi}{12}}$

Put $k=2$ $x_2 = 2\sqrt{2} e^{i\frac{17\pi}{12}}$

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The continued product of roots is given by ,

$$\begin{aligned} x_0 x_1 x_2 &= 2\sqrt{2}e^{i\frac{\pi}{12}} \times 2\sqrt{2}e^{i\frac{9\pi}{12}} \times 2\sqrt{2}e^{i\frac{17\pi}{12}} \\ &= 16\sqrt{2}e^{i\frac{27\pi}{12}} \\ &= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \\ &= 1+i \end{aligned}$$

The continued product of roots is (1+i).

(b) Find non singular matrices P & Q such that PAQ is in normal form

Where $A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$ [6]

Ans : Matrix in PAQ form is given by ,

$$A = P A Q$$

$$\begin{bmatrix} 2 & -2 & 3 \\ 3 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \rightarrow R_3,$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - 3R_1, R_3 - 2R_1,$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 - 2C_1, C_3 + C_1,$$

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$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\frac{C_3}{5}$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 1 \\ 0 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 1/5 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$C_2 + 6C_3$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -4/5 & 1/5 \\ 0 & 1 & 0 \\ 0 & 6/5 & 1/5 \end{bmatrix}$$

$C_3 + C_2$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -4/5 & -3/5 \\ 0 & 1 & 1 \\ 0 & 6/5 & 7/5 \end{bmatrix}$$

$-R_1$,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -4/5 & -3/5 \\ 0 & 1 & 1 \\ 0 & 6/5 & 7/5 \end{bmatrix}$$

Now A is in normal form with rank 3.

Compare with PAQ form,

$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 0 & -2 \end{bmatrix}$	$Q = \begin{bmatrix} 1 & -4/5 & -3/5 \\ 0 & 1 & 1 \\ 0 & 6/5 & 7/5 \end{bmatrix}$
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(c) Find the maximum and minimum values of

$$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x \quad [8]$$

Ans: given: $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

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$$f_x = 3x^2 + 3y^2 - 30x + 72 \quad f_{xx} = 6x - 30$$

$$f_y = 6xy - 30y \quad f_{yy} = 6x - 30$$

$$f_{xy} = 6y$$

To find stationary values :

$$f_x = 3x^2 + 3y^2 - 30x + 72 = 0 \quad \& \quad f_y = 6xy - 30y = 0$$

$$y=0 \text{ or } x=5$$

for $y=0$, $x=6,4$

$$\therefore (x,y)=(6,0) , (4,0).$$

For $x=5$, $y=1,-1$

$$\therefore (x,y)=(5,1) , (5,-1)$$

Stationary points are : $(6,0),(4,0),(5,1),(5,-1)$

(i) For point $(6,0)$,

$$r = f_{xx} = 36 - 30 = 6 , \quad s = f_{xy} = 0 , \quad t = f_{yy} = 6$$

$$rt - s^2 = 36 > 0 \quad \text{and } r = 6 > 0$$

function is minimum at $(6,0)$.

$$f_{min} = 108$$

(ii) For point $(4,0)$,

$$r = f_{xx} = -6 , \quad s = f_{xy} = 0 , \quad t = f_{yy} = -6$$

$$rt - s^2 = 36 > 0 \quad \text{and } r = -6 < 0$$

function is maximum at $(4,0)$.

$$f_{max} = 112$$

(iii) for point $(5,1)$ and $(5,-1)$,

Thr points are neither maximum nor minimum.

\therefore The maximum and minimum value of function are 112 and 108 .

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Q.3(a) If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, show that $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$. [6]

Ans : let $u = f(r, s)$

$$\therefore r = \frac{y-x}{xy} \qquad \therefore s = \frac{z-x}{xz}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} = \frac{\partial u}{\partial r} \frac{1}{x^2} + \frac{\partial u}{\partial s} \left(\frac{-1}{x^2}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial y} = \frac{\partial u}{\partial r} \frac{(-1)}{y^2} + \frac{\partial u}{\partial s} (0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial z} = \frac{\partial u}{\partial r} (0) + \frac{\partial u}{\partial s} \left(\frac{1}{z^2}\right)$$

$$\therefore x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$$

Hence proved.

(b) Using encoding matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, encode & decode the message

“MUMBAI”. [6]

Ans : Encoding matrix : $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Message is : MUMBAI.

The given message in matrix form is :

$$B = \begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix}$$

Encoded message in matrix form is given by ,

$$C = A.B$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} 34 & 15 & 10 \\ 21 & 2 & 9 \end{bmatrix}$$

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Encoded message is : 34 21 15 2 10 9

GUOJJI

Decoded matrix is given by ,

$$\begin{aligned} B &= A^{-1} \cdot C \\ &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 34 & 15 & 10 \\ 21 & 2 & 9 \end{bmatrix} \\ \therefore B &= \begin{bmatrix} 13 & 13 & 1 \\ 21 & 2 & 9 \end{bmatrix} \end{aligned}$$

Decoded message : MUMBAI

(c) Prove that $\log\left[\tan\left(\frac{\pi}{4} + \frac{ix}{2}\right)\right] = i \tan^{-1}(\sinh x)$ [8]

Ans : L.H.S = $\log\left[\tan\left(\frac{\pi}{4} + \frac{ix}{2}\right)\right]$

$$\begin{aligned} &= \log\left[\frac{1 + \tan\left(\frac{ix}{2}\right)}{1 - \tan\left(\frac{ix}{2}\right)}\right] \\ &= \log\left[1 + \tan\left(\frac{ix}{2}\right)\right] - \log\left[1 - \tan\left(\frac{ix}{2}\right)\right] \\ &= \log\left[1 + i \tanh\frac{x}{2}\right] - \log\left[1 - i \tanh\frac{x}{2}\right] \end{aligned}$$

We have ,

$$\log(a+ib) = \frac{1}{2} \log(a^2 + b^2) + i \tan^{-1}\left(\frac{b}{a}\right)$$

$$\begin{aligned} \therefore &= \frac{1}{2} \log\left(1 + \tanh^2\frac{x}{2}\right) + i \tan^{-1}\left(\tanh\frac{x}{2}\right) - \left[\frac{1}{2} \log\left(1 + \tanh^2\frac{x}{2}\right) - i \tan^{-1}\left(\tanh\frac{x}{2}\right)\right] \\ &= 2i \left[\tan^{-1}\left(\tanh\frac{x}{2}\right)\right] \end{aligned}$$

$$\text{L.H.S} = 2i \cdot \tan^{-1}\left(\tanh\frac{x}{2}\right)$$

$$\text{R.H.S} = i \cdot \tan^{-1}(\sinh x)$$

We know that $\sinh^{-1}x = \log(x + \sqrt{1 + x^2})$

$$\tanh^{-1}x = \frac{1}{2} \left[\log\left(\frac{x+1}{1-x}\right)\right]$$

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$$= i \tan^{-1} \left(\tanh \frac{x}{2} \right)$$

Also $\sinh^{-1}(\tan x) = \tanh^{-1}(x)$

R.H.S = $i \cdot \tan^{-1} \left(\tanh \frac{x}{2} \right)$

$$\log \left[\tan \left(\frac{\pi}{4} + \frac{ix}{2} \right) \right] = i \cdot \tan^{-1}(\sinh x)$$

Q.4(a) Obtain $\tan 5\theta$ in terms of $\tan \theta$ & show that

$$1 - 10 \tan^2 \frac{x}{10} + 5 \tan^4 \frac{x}{10} = 0 \quad [6]$$

Ans: we have $\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

Put $n=5$,

$$\begin{aligned} \therefore \cos 5\theta + i \sin 5\theta &= (\cos \theta + i \sin \theta)^5 \\ &= \cos^5 \theta + 5 \cos^4 \theta \cdot i \sin \theta + 10 \cos^3 \theta \cdot (i \sin \theta)^2 \\ &\quad + 10 \cos^2 \theta \cdot (i \sin \theta)^3 + 5 \cos \theta \cdot (i \sin \theta)^4 + i \sin^5 \theta \\ &= [\cos^5 \theta - 10 \cos^3 \theta \cdot (\sin \theta)^2 \\ &\quad + 5 \cos \theta \cdot (\sin \theta)^4] + [5 \cos^4 \theta \cdot i \sin \theta \\ &\quad - 10 i \cos^2 \theta \cdot (\sin \theta)^3 + i \sin^5 \theta] \end{aligned}$$

Compare real and imaginary parts

$$\cos 5\theta = [\cos^5 \theta - 10 \cos^3 \theta \cdot (\sin \theta)^2 + 5 \cos \theta \cdot (\sin \theta)^4]$$

$$\sin 5\theta = +[5 \cos^4 \theta \cdot \sin \theta - 10 \cos^2 \theta \cdot (\sin \theta)^3 + \sin^5 \theta]$$

$$\tan 5\theta = \frac{[5 \cos^4 \theta \cdot \sin \theta - 10 \cos^2 \theta \cdot (\sin \theta)^3 + \sin^5 \theta]}{[\cos^5 \theta - 10 \cos^3 \theta \cdot (\sin \theta)^2 + 5 \cos \theta \cdot (\sin \theta)^4]}$$

$$\tan 5\theta = \frac{5 \tan \theta - 10 \tan^3 \theta + \tan^5 \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}$$

put $\theta = \frac{\pi}{10}$

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$$1 - 10 \tan^2 \frac{x}{10} + 5 \tan^4 \frac{x}{10} = 0$$

(b) If $y = e^{\tan^{-1} x}$. Prove that

$$(1 + x^2)y_{n+2} + [2(n + 1)x - 1]y_{n+1} + n(n + 1)y_n = 0 \quad [6]$$

Ans : $y = e^{\tan^{-1} x} \quad \dots\dots(1)$

Diff. w.r.t x,

$$y_1 = e^{\tan^{-1} x} \frac{1}{x^2 + 1}$$

$$(x^2 + 1)y_1 = e^{\tan^{-1} x} = y \quad \text{-----(from 1)}$$

Again diff. w.r.t x,

$$(x^2 + 1)y_2 + 2xy_1 = y_1 \quad \dots\dots\dots(1)$$

Now take n th derivative by applying Leibnitz theorem,

Leibnitz theorem is :

$$(uv)_n = u_n v + {}^n C_1 u_{n-1} v_1 + {}^n C_2 u_{n-2} v_2 + \dots + u v_n$$

$$u = (x^2 + 1), v = y_2 \quad \dots \text{for first term in eqn (1)}$$

$$u = 2x, v = y_1 \quad \dots \text{for second term in eqn (1)}$$

$$\therefore (1 + x^2)y_{n+2} + 2(n + 1)xy_{n+1} + n(n + 1)y_n - y_{n+1} = 0$$

$$\therefore (1 + x^2)y_{n+2} + [2(n + 1)x - 1]y_{n+1} + n(n + 1)y_n = 0$$

Hence Proved.

(c) i. Express $(2x^3 + 3x^2 - 8x + 7)$ in terms of $(x-2)$ using Taylor's

Series.

[4]

ii. Prove that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$ [4]

Ans: i. let $f(x) = 2x^3 + 3x^2 - 8x + 7$

Here $a = 2$

$$f(x) = 2x^3 + 3x^2 - 8x + 7 \quad f(2) = 19$$

$$f'(x) = 6x^2 + 6x - 8 \quad f'(2) = 28$$

$$f''(x) = 12x + 6 \quad f''(2) = 30$$

$$f'''(x) = f'''(2) = 12$$

Taylor's series is :

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots$$

$$2x^3 + 3x^2 - 8x + 7 = 19 + (x - 2)28 + \frac{(x-2)^2}{2!}30 + \frac{(x-a)^3}{3!}12$$

$$2x^3 + 3x^2 - 8x + 7 = 19 + 28(x - 2) + 15(x - 2)^2 + 2(x - 2)^3$$

ii. let $y = \tan^{-1} x$

diff. w.r.t x ,

$$\therefore y_1 = \frac{1}{x^2+1}$$

Series expansion of y_1 ,

We know that,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$\therefore y_1 = 1 - x^2 + x^4 - x^5$$

Integrate y_1 to find series expansion of y ,

$$\therefore y = \int (1 - x^2 + x^4 - x^5 + \dots) dx$$

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$$\therefore y = x \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

Hence Proved .

Q.5(a) If $z = x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \frac{x}{y}$ ∂

Prove that $\frac{\partial^2 z}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$ [6]

Ans : $z = x^2 \tan^{-1} \frac{y}{x} - y^2 \tan^{-1} \frac{x}{y}$

Diff. w.r.t. x partially ,

$$\begin{aligned} \frac{\partial z}{\partial x} &= x^2 \frac{x^2}{x^2 + y^2} \times \frac{-y}{x^2} + \tan^{-1} \frac{y}{x} \cdot 2x - y^2 \frac{y^2}{x^2 + y^2} \times \frac{1}{y} \\ &= \frac{x^2}{x^2 + y^2} \times \frac{-y}{1} + 2x \tan^{-1} \frac{x}{y} - \frac{y^3}{x^2 + y^2} \end{aligned}$$

Diff. w.r.t y partially ,

$$\begin{aligned} \frac{\partial^2 z}{\partial y \partial x} &= -x^2 \left[-y \cdot \frac{2y}{(x^2 + y^2)^2} + \frac{1}{x^2 + y^2} \right] + 2 \frac{x^2}{x^2 + y^2} - \left[-y^3 \cdot \frac{2y}{(x^2 + y^2)^2} + \frac{3y^2}{x^2 + y^2} \right] \\ &= \left[\frac{2y^3 x^2}{(x^2 + y^2)^2} + \frac{-x^2}{x^2 + y^2} \right] + 2 \frac{x^2}{x^2 + y^2} + \frac{2y^4}{(x^2 + y^2)^2} - \frac{3y^2}{x^2 + y^2} \\ &= \frac{(x^2 - y^2)^2 \times (x^2 + y^2)^1}{(x^2 + y^2)^2 (x^2 - y^2)^1} \\ &= \frac{x^2 - y^2}{x^2 + y^2} \end{aligned}$$

$$\therefore \frac{\partial^2 z}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$$

Hence proved.

(b) Investigate for what values of μ and λ the equations : $2x + 3y + 5z = 9$

$$7x + 3y - 2z = 8$$

$$2x + 3y + \lambda z = \mu$$

Have (i) no solution (ii) unique solution (iii) Infinite value [6]

Ans : Given eqn : $2x + 3y + 5z = 9$

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$$7x+3y-2z=8$$

$$2x+3y+\lambda z=\mu$$

$$A X = B$$

$$\therefore \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

Argumented matrix is : $\left[\begin{array}{ccc|c} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & \lambda & \mu \end{array} \right]$

$$R_3 - R_1,$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 5 & 6 \\ 7 & 3 & -2 & 4 \\ 0 & 0 & \lambda - 5 & \mu - 9 \end{array} \right]$$

(i) When $\lambda=5, \mu \neq 9$ then $r(A) = 2, r(A : B) = 3$

$$r(A) \neq r(A : B)$$

No Solution.

(ii) When $\lambda \neq 5, \mu \neq 9, r(A) = r(A : B) = 3$

Unique solution exist.

(iii) When $\lambda=5, \mu = 9, r(A) = r(A : B) = 2 < 3$

Infinite solution.

(c) Obtain the root of $x^3 - x - 1 = 0$ by Newton Raphson Method

(upto three decimal places).

[8]

Ans : Equation : $x^3 - 2x - 5 = 0$

$$\therefore f(x) = x^3 - 2x - 5$$

$$f(0) = -5 < 0 \text{ and } f(1) = -2 < 0 \text{ and } f(2) = 7 > 0.$$

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Root of given eqn lies between 1 and 2.

$$f'(x) = 3x^2 + 2$$

Let take $x_0 = 2$

$$\begin{aligned}x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 2 - \frac{7}{14} = 1.5\end{aligned}$$

Next iteration :

$$\begin{aligned}\therefore x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 1.343 \\ \therefore x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 1.329\end{aligned}$$

For next iteration :

$$\begin{aligned}\therefore x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} = 1.329 - \frac{f(1.329)}{f'(1.329)} \\ &= 1.3283\end{aligned}$$

The root of eqn is $x = 1.3283$

Q.6(a) Find $\tanh x$ if $5\sinh x - \cosh x = 5$

[6]

Ans : $5\sinh x - \cosh x = 5$

But $\sinh x = \frac{e^x - e^{-x}}{2}$ $\cosh x = \frac{e^x + e^{-x}}{2}$

$$\therefore 5\left[\frac{e^x - e^{-x}}{2}\right] - \left[\frac{e^x + e^{-x}}{2}\right] = 5$$

$$\therefore 5e^x - 5e^{-x} - e^x - e^{-x} = 10$$

$$\therefore 4e^{2x} - 10e^x - 6 = 0$$

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Roots are : $e^x = 3, e^x = \frac{-1}{2}$

$$\therefore \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\left(\frac{-1}{2}\right) + 2}{-5/2} = \frac{-3}{5}$$

Or

$$\therefore \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{3 - 1/3}{3 + 1/3} = \frac{4}{5}$$

The values of $\tanh x$ are : $\frac{-3}{5}$ or $\frac{4}{5}$

(b) If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, Prove that i. $xu_x + yu_y = \frac{1}{2} \tan u$

ii. $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{-\sin u \cos 2u}{4\cos^3 u}$ [6]

Ans : $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

Put $x = xt$ and $y = yt$ to find degree.

$$\therefore u = \sin^{-1}\left(\frac{xt+yt}{\sqrt{xt}+\sqrt{yt}}\right)$$

$$\therefore \sin u = t^{1/2} \cdot \frac{x+y}{\sqrt{x}+\sqrt{y}} = t^{1/2} \cdot f(x, y)$$

The function $\sin u$ is homogeneous with degree $\frac{1}{2}$.

But $\sin u$ is the function of u and u is the function of x and y .

By Euler's theorem ,

$$xu_x + yu_y = G(u) = n \cdot \frac{f(u)}{f'(u)} = \frac{1}{2} \tan u$$

$$\therefore xu_x + yu_y = \frac{1}{2} \tan u$$

$$\therefore x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = G(u)[G'(u) - 1]$$

$$= \frac{1}{2} \tan u \left[\frac{\sec^2 u - 2}{2} \right]$$

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$$= \frac{1}{4} \tan u \left[\frac{\tan^2 u - 1}{1} \right]$$

$$= \frac{1}{4} \times \frac{\sin u}{\cos u} \left[\frac{\sin^2 u - \cos^2 u}{\cos^2 u} \right]$$

$$\therefore x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = \frac{-\sin u \cdot \cos 2u}{4 \cos^3 u}$$

Hence Proved.

(c) Solve the following system of equation by Gauss Siedal Method,

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

[8]

Ans: By Gauss Seidal method ,

Given eqn : $20x + y - 2z = 17$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

From given eqn : $|20| > |1| + |-2|$

$$|20| > |3| + |-1|$$

$$|20| > |2| + |-3|$$

The given eqn are in correct order.

$$\therefore x = \frac{1}{20} [17 - y + 2z]$$

$$\therefore y = \frac{1}{20} [-18 - 3x + z]$$

$$\therefore z = \frac{1}{20} [25 - 2x + 3y]$$

i) For 1st iteration : take $y = 0, z = 0$

$$x = \frac{1}{20} [17] = 0.85$$

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$$x = 0.85, z = 0 \quad \text{gives } y = -1.0275$$

$$x = 0.85, y = -1.0275 \quad \text{gives } x_3 = 1.0109$$

II) For 2nd iteration : take $y = -1.0275, z = 1.0109$

$$x = \frac{1}{20} [17 + 1.0275 - 2(1.0109)] = 1.0025$$

$$x = 1.0025, z = 1.0109 \quad \text{gives } y = -0.9998$$

$$x = 1.0025, y = -0.9998 \quad \text{gives } z = 0.9998$$

III) For 3rd iteration : $y = -0.9998, z = 0.9998$

$$x_1 = \frac{1}{20} [17 + 0.9998 + 2(0.9998)] = 1.00$$

$$x = 1.00, z = 0.9998 \quad \text{gives } y = -1.00$$

$$x = 1.00, y = -1.00 \quad \text{gives } z = 1.00$$

Result : $x = 1.00, y = -1.00, z = 1.00$

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